

Uniqueness results for critical points of a non-local isoperimetric problem via curve shortening

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Abstract

Using area-preserving curve shortening flow, and a new inequality relating the potential generated by a set to its curvature, we study a non-local isoperimetric problem which arises in the study of di-block copolymer melts, also referred to as the Ohta-Kawasaki energy. We are able to show that the only connected critical point is the ball under mild assumptions on the boundary, in the small energy/mass regime. In particular this class includes all rectifiable, connected 1-manifolds in \mathbb{R}^2 . We also classify the simply connected critical points on the torus in this regime, showing the only possibilities are the stripe pattern and the ball. In \mathbb{R}^2 , this can be seen as a partial union of the well known result of Fraenkel [19] for uniqueness of critical points to the Newtonian Potential energy, and Alexandrov for the perimeter functional [2], however restricted to the plane. The proof of the result in \mathbb{R}^2 is analogous to the curve shortening result due to Gage [22], but involving a non-local perimeter functional, as we show the energy of convex sets strictly decreases along the flow. Using the same techniques we obtain a stability result for minimizers in \mathbb{R}^2 and for the stripe pattern on the torus, the latter of which was recently shown to be the global minimizer to the energy when the non-locality is sufficiently small [46].

1 Introduction

The classical isoperimetric problem has been thoroughly studied, and it is well known since the work of De Giorgi [16] that the unique optimizer to this problem in \mathbb{R}^d is the ball. There has recently been significant interest in the effects of adding a repulsive term to the classic perimeter which favors separation of mass. An example of such an energy is often referred to as the Ohta-Kawasaki energy, first introduced in [39], and takes the following form

$$E[u] := \int_U |\nabla u| + \gamma \int_U \int_U (u(x) - \bar{u})G(x, y)(u(y) - \bar{u})dxdy, \quad (1)$$

where $u \in BV(U; \{0, +1\})$ and

$$\bar{u} = \begin{cases} \int_U u(x)dx & \text{for } U \text{ bounded and open} \\ 0 & \text{for } U = \mathbb{R}^d. \end{cases} \quad (2)$$

Here $BV(U; \{0, +1\})$ denotes the space of functions of bounded variation taking values 0 and +1 in U (see [3] for an introduction to the space BV). The kernel G is generally taken to be the kernel of the Laplacian operator, with periodic boundary conditions when $U = \mathbb{T}^d$. The parameter $\gamma > 0$ describes the strength of the non-local term. It is clear that the non-local term favors the separation of mass while the perimeter favors clustering. The above problem describes a number of polymer systems [15, 38, 42] as well as many other physical systems [7, 17, 24, 30, 29, 38] due to the fundamental nature of the Coulombic term. Despite the abundance of physical systems for which (1) is applicable, rigorous mathematical analysis is fairly recent [1, 8, 9, 10, 11, 12, 13, 25, 26, 32, 33, 34, 35, 36]. One considers minimizing (1) over the class

$$\mathcal{A}_m := \{u \in BV(U; \{0, +1\}) : \int_U u dx = m\}, \quad u := \chi_\Omega. \quad (3)$$

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The variational problem (1) when $U = \mathbb{T}^2$ has been studied in the limit $\gamma \rightarrow +\infty$ when the phase $\{u = 0\}$ dominates the $\{u = +1\}$ [12, 13, 26, 25, 34]. The minimizers in this case form droplets of the minority phase $\{u = +1\}$ in the majority phase $\{u = 0\}$ and each connected component of $\{u = +1\}$ wishes to minimize the energy

$$E_{\mathbb{R}^d}[u] = \int_{\mathbb{R}^d} |\nabla u| + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x)G(x, y)u(y)dx dy, \quad (4)$$

over $u \in \mathcal{A}_m$, where G is a kernel on \mathbb{R}^d to be explicitied below and m is $O(1)$ as $\gamma \rightarrow +\infty$. When we refer to (1) with $U = \mathbb{R}^d$ we will mean (4) throughout the paper. Observe that we no longer have the constant γ in front of the non-local term in this case, as one can rescale the domain Ω to make $\gamma \equiv 1$ for the class of G we consider (see (13) below), changing the energy by at most a constant. Indeed when $\gamma \neq 1$, by a change of variables $x' = \lambda x$, $y' = \lambda y$ and setting $\tilde{u}(x) = u(x/\lambda)$ we can write (1) as

$$E[u] = \lambda \left(\int_{\mathbb{R}^d} |\nabla \tilde{u}| + \gamma \lambda^{3-p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{u}(x)G(x, y)\tilde{u}(y)dx dy + C(\lambda) \right), \quad (5)$$

where $C(\lambda) > 0$ is a constant which arises when G is the logarithmic kernel, and $p > 0$ arises from the singularity of the kernel. Setting $\lambda^{3-p}\gamma = 1$ yields (4) after dividing by λ and subtracting the constant term. We will often abuse terminology and say $\Omega \in \mathcal{A}_m$ when we mean $\chi_\Omega \in \mathcal{A}_m$ and write $E(\Omega)$ when we mean $E(\chi_\Omega)$. For the remainder of the paper we will study (1) exclusively with $U = \mathbb{T}^2$ and $U = \mathbb{R}^2$, except for comments regarding results in \mathbb{R}^d for $d \geq 3$.

When $U = \mathbb{R}^2$, in order for minimizers to exist in \mathcal{A}_m to (1), it is necessary to modify the logarithmic kernel. The above energy with G replaced by the kernel

$$K(x) = \frac{1}{|x|^\alpha} \text{ for } \alpha \in (0, 2), \quad (6)$$

was recently studied by Knupfer and Muratov [35, 36]. A simple rescaling shows that for small masses, the effect of the non-local term is small compared to that of the perimeter. It is therefore reasonable to expect that for small masses, the unique minimizer to (1) is the ball. This was shown by Knupfer and Muratov [35, 36] in dimensions $2 \leq d \leq 8$. Moreover they show that for sufficiently large masses, for $2 \leq d \leq 8$, minimizers of (1) fail to exist, as it is favorable for mass to split.

There is also much interest in critical points to (1) which are not necessarily locally minimizing. Here a critical point to (1) is a set $\Omega \in \mathcal{A}_m$ for which the first variation with respect to volume preserving diffeomorphisms vanish.

Definition 1. A set $\Omega \in \mathcal{A}_m$ is a critical point of (1) if for all volume preserving diffeomorphisms $\phi_t : \Omega \rightarrow \phi_t(\Omega) =: \Omega_t$ it holds that

$$\left. \frac{dE(\Omega_t)}{dt} \right|_{t=0} = 0. \quad (7)$$

In particular a simple calculation [14] shows that Ω is a smooth critical point if and only if it solves the following Euler-Lagrange equation

$$\kappa(y) + \gamma \phi_\Omega(y) = \lambda \text{ on } \partial\Omega \quad (8)$$

where ϕ_Ω is the potential generated by Ω , ie.

$$\phi_\Omega(y) = \begin{cases} \int_\Omega G(x, y)dx & \text{for } U = \mathbb{R}^2 \\ \int_{\mathbb{T}^2} G(x, y)(u(x) - \bar{u})dx & \text{for } U = \mathbb{T}^2, \end{cases} \quad (9)$$

κ is the curvature of $\partial\Omega$ and λ is the Lagrange multiplier arising from the volume constraint. We however do not wish to assume a-priori regularity of the boundary as critical points may in general not be smooth. An important example demonstrating this is the coordinate axes in \mathbb{R}^2 minus the origin. In this case the generalized mean curvature is constant on the reduced boundary $\partial^*\Omega = \Omega$, $m = 0$ and hence (8) is satisfied everywhere on $\partial^*\Omega$ (see [44] for a reference on generalized mean curvature and the theory of

varifolds). The example of a figure 8 with center at O also shows that while the curvature of a closed curve can be smooth and uniformly bounded on $\partial\Omega \setminus \{O\} = \partial^*\Omega$, there is not necessarily a natural way to make sense of the curvature or variations of (1) near O . In \mathbb{R}^2 however we can continue to make sense of the curvature at O if there exists a parametrization of the boundary by a closed, rectifiable (ie. has finite length) curve. By the results of [4], any connected set with finite perimeter has a boundary $\partial\Omega$ which can be decomposed into a countable union of Jordan curves $\{\gamma_k\}_k$ with disjoint interiors. In this case however, as the figure 8 example demonstrates, one cannot make sense of variations near points on the curve which are locally homeomorphic to $[0, 1]$, and can thus only expect to extract information from the Euler-Lagrange information on the reduced boundary. However, when the boundary $\partial\Omega$ can be decomposed into a countable *disjoint* union of closed, rectifiable curves, there is a natural way to consider variations of the domain, even on the complement of the reduced boundary, by considering variations of the curve in the normal direction induced by the parametrization. We thus make the following definition.

Definition 2. (*Admissible curves*) A connected set Ω with finite perimeter will be called admissible if its boundary $\partial\Omega$ can be decomposed into a countable number of closed, disjoint curves γ_k each admitting a $W^{1,1}$ parametrization $\gamma_k : [0, 1] \rightarrow \mathbb{R}^2$ with $|\gamma'_k(t)| = L_k$ for $t \in [0, 1]$. In particular we may write $\gamma_k(t) = \gamma_k(0) + L_k \int_0^t e^{i\theta_k(r)} dr$.

Note that the above class includes all connected, rectifiable 1-manifolds. Indeed any manifold has a boundary which is not locally homeomorphic to $[0, 1]$ and thus does not intersect any other segments of the boundary. In particular, each γ_k is therefore simple and disjoint from every other γ_j for $j \neq k$, and never intersects itself transversally. Working within the class of admissible curves, we are able to rigorously compute the Euler-Lagrange equation and extract sufficient information from it to conclude that admissible critical points are convex when $U = \mathbb{R}^2$ and simply connected critical points are convex or star shaped when $U = \mathbb{T}^2$, in the small mass/energy regime. The details will be presented in Section 2.

When $U = \mathbb{R}^2$, the classification of critical points corresponding to either the perimeter term or non-local term, considered separately, has been well studied [2, 19]. In particular it is a well known result of Alexandrov [2] that in dimensions $d \geq 2$ the only simply connected, compact, constant mean curvature surface is the ball. For the non-local term, Fraenkel showed somewhat recently [19] that if $\phi_\Omega \equiv \text{constant}$ on $\partial\Omega$ then Ω must be the ball. This was recently extended to general Riesz kernels by Reichel [40], however restricted to the class of convex sets. The question then naturally arises of knowing how one can classify the solutions to (8). In \mathbb{R}^2 one can easily construct annuli which satisfy (8) for particular choices of radii (see Counter Example 1). Even in \mathbb{R}^3 examples of tori and double tori solutions to (8) exist [47], showing that compact, connected solutions to (8) exist other than the ball. Since a smooth set is a critical point in the sense of Definition 1 if and only if it satisfies 8 (see [14]), this equivalently shows a lack of uniqueness for critical points of (1). We provide a partial answer to this question (see Theorem 1 below) for a range of values of $(m, E) \in \mathbb{R}^+ \times \mathbb{R}^+$ sufficiently small, by showing that the only connected critical point is the ball in the class of admissible sets in that range. More precisely, when the parameters (14) or (15) are sufficiently small.

It is easily seen that the only constant curvature surfaces in \mathbb{T}^2 are unions of circular arcs and straight lines. In addition, the stripe patterns defined by

$$u_n(x) = u(nx) \text{ for } n \in \mathbb{N}, \quad (10)$$

where

$$u(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq w \\ 0 & \text{for } w \leq x \leq 1 \end{cases}, \quad (11)$$

for $w \in (0, 1)$ also satisfy $\phi_\Omega = \text{constant}$ on ∂S_n where S_n is the set corresponding to the indicator function u_n . We are also able to classify simply connected solutions to (8) in this case, showing that in fact, for a range of $(\gamma, E) \in \mathbb{R}^+ \times \mathbb{R}^+$ sufficiently small, there are no solutions to (8) other than $S_{n=1}$ and the ball. In particular, when $m > \frac{\pi}{4}$, the only possibility is $S_{n=1}$.

The way that we characterize critical points in both cases is by showing that any set Ω as described above which is not a constant curvature surface satisfies

$$\left. \frac{dE(\Omega_t)}{dt} \right|_{t=0} \neq 0, \quad (12)$$

where Ω_t is the evolution of Ω under area-preserving curve shortening flow, which is admissible under Definition 1. Details will follow in Sections 3 and 4.

When $U = \mathbb{R}^2$ our main results (Theorems 1 and 3 below) will hold for

$$G(x, y) = -\frac{1}{2\pi} \log |x - y| \text{ or } G(x, y) = \frac{1}{|x - y|^\alpha} \text{ when } \alpha \in (0, 1), \quad (13)$$

with minor modifications to the proofs. It will always be made clear below which kernel is being used. When a constant depends on α , this will mean exclusively for the kernel K for $\alpha \in (0, 1)$. In all such cases the dependence of the constant on α may be dropped for the logarithmic kernel.

We begin by defining the following parameters

$$\bar{\eta} := \begin{cases} m^{1/2} L^2 (1 + |\log L|) & \text{for } G(x, y) = -\frac{1}{2\pi} \log |x - y|, \quad \gamma \equiv 1 \\ m^{1/2} L^{2-\alpha} & \text{for } G(x, y) = \frac{1}{|x - y|^\alpha} \quad \alpha \in (0, 1), \quad \gamma \equiv 1 \end{cases} \quad (14)$$

$$\bar{\gamma} := \gamma m^{1/2} L^2 (1 + |\log L|), \quad m \in (0, 1) \quad (15)$$

where $L = |\partial\Omega|$. Our results will be stated in terms of these rescaled parameters. A simple scaling analysis of (1) reveals

$$\left\langle \frac{d(E - L)}{dL}, \zeta \right\rangle = \gamma \frac{\int_{\partial\Omega} \phi_\Omega(y) \zeta(y) dS(y)}{\int_{\partial\Omega} \kappa(y) \zeta(y) dS(y)} \sim \bar{\eta}, \bar{\gamma}, \quad (16)$$

where E is defined by (1), dS is surface measure on $\partial\Omega$ and with some abuse of notation $\left\langle \frac{d(E - L)}{dL}, \zeta \right\rangle$ denotes the variation in the sense of Definition 1 induced by the normal velocity $\zeta : \partial\Omega \rightarrow \mathbb{R}$. Thus $\bar{\eta}$ and $\bar{\gamma}$ represent the rate of change of the non-local term in the energy with respect to the length of the boundary. Our result can thus be stated formally as saying that when the change of the non-local term is small compared to a change in length, the critical points can be classified entirely in terms of those of the length term in (1), and thus are constant curvature curves.

For minimizers we have a natural a priori bound on L coming from the positivity of both terms in the energy (1), which we don't have for non-minimizing critical points. This explains the need for introducing (14)–(15). The terms $\bar{\eta}_{cr}$ and $\bar{\gamma}_{cr}$ below are critical values of $\bar{\eta}$ and $\bar{\gamma}$ which can be made explicit and are described in more detail in Section 4.

Theorem 1. *When $U = \mathbb{R}^2$ there exists $\bar{\eta}_{cr} = \bar{\eta}_{cr}(\alpha) > 0$ such that whenever $\bar{\eta} < \bar{\eta}_{cr}$, the only admissible critical point of (1) in \mathcal{A}_m in the sense of Definition 2 is the ball. When $U = \mathbb{T}^2$ there exists a $\bar{\gamma}_{cr} > 0$ such that whenever $\bar{\gamma} < \bar{\gamma}_{cr}$, the only simply connected critical points to (1) in \mathcal{A}_m for all $m \in (0, 1]$ are the ball and the stripe pattern $S_{n=1}$ defined by (10)–(11). In particular when $m > \frac{\pi}{4}$, the only simply connected critical point is $S_{n=1}$.*

The first part of the Theorem when $U = \mathbb{R}^2$ is false when the mass is larger, as annuli satisfying (8) can easily be constructed.

Counter Example 1. *There exists a smooth, compact, connected set Ω solving (8) with $\bar{\eta} > \bar{\eta}_{cr}$ which is not the ball.*

Remark 1. *It is easy to see that there cannot exist critical points with multiple disjoint components which are separable by a hyperplane. Indeed if Ω_1 and Ω_2 are two disjoint components of Ω which can be separated, then let c be a vector so that $(x - y) \cdot c > 0$ for $(x, y) \in \Omega_1 \times \Omega_2$. Then we have*

$$\frac{d}{dt} \iint_{\Omega_1 \times \Omega_2} \log |x + ct - y| dx dy = - \iint_{\Omega_1 \times \Omega_2} \frac{(x - y)}{|x - y|^2} \cdot c dx dy < 0. \quad (17)$$

Consequently Ω cannot be critical in the sense of Definition 1. Theorem 1 however leaves open the possibility of more intricate solutions to (8), with multiple connected components. A similar calculation shows the same result for the kernel $K(x) = 1/|x|^\alpha$.

Using the same techniques we also obtain the following stability results. The first concerns global minimizers on the torus \mathbb{T}^2 and can be seen as a statement about the stability of the recent result of Sternberg and Topaloglu [46]. We consider the class of connected sets belonging to \mathcal{A}_m

$$\mathcal{A}_m^c = \{u \in \mathcal{A}_m : \Omega \text{ is simply connected}\}. \quad (18)$$

We then have the following theorem. Note that we use the original parameter γ .

Theorem 2. *When $m = \frac{1}{2}$ and $U = \mathbb{T}^2$, there exists a $\gamma_{cr} > 0$, a functional $F : \mathcal{A}_{\frac{1}{2}}^c \rightarrow \mathbb{R}$ and $C = C(\gamma_{cr}) > 0$ such that whenever $\gamma < \gamma_{cr}$*

$$E(\Omega) \geq E(S_{n=1}) + CF(\Omega), \quad (19)$$

where $F(\Omega) \geq 0$ with equality if and only if $\Omega = S_{n=1} := [0, \frac{1}{2}] \times [0, 1]$.

The above theorems all rely on Theorem 10 in Section 4 which provides an explicit estimate for the rate of decrease of the energy (1) along area-preserving curve shortening flow. The following stability result for minimizers in \mathbb{R}^2 is a simple corollary of Theorem 10.

Corollary 1. *Let Ω be any convex set in \mathbb{R}^2 . Then there exists an $m_{cr} > 0$ such that whenever $m < m_{cr}$ there is a constant $C = C(m_{cr}) > 0$ such that*

$$E(\Omega) \geq E(B) + C(L - 2\sqrt{\pi}m^{1/2}),$$

where $L = |\partial\Omega|$.

The main geometric inequality which we prove in this paper in order to control the non-local term along the flow is the following.

Theorem 3. *Let $\Omega \subset U$ be a convex set with $\kappa \in L^2(\partial\Omega)$ and $U = \mathbb{R}^2$ or $U = \mathbb{T}^2$. Then there is an explicit constant $C = C(|\partial\Omega|, \alpha) > 0$ such that*

$$\|\phi_\Omega - \bar{\phi}_\Omega\|_{L^\infty(\partial\Omega)}^2 \leq C \int_{\partial\Omega} |\kappa - \bar{\kappa}|^2 dS,$$

where the bar denotes the average over $\partial\Omega$ and dS is 1-dimensional Hausdorff measure. The above continues to hold in \mathbb{T}^2 for any set homeomorphic to $S_{n=1}$.

The above theorem provides a quantitative estimate of the closeness to an equipotential surface in terms of the distance to a constant curvature surface. The inequality in \mathbb{R}^2 in fact relies on an isoperimetric inequality due to Gage [22] applied to curve shortening for convex sets. We hope that the above inequality will be of interest even outside the context of Ohta-Kawasaki.

Remark 2. *We remark that if Ω is any connected set with $\kappa \in L^2(\partial\Omega)$ we can prove the weaker inequality*

$$\|\phi_\Omega - \bar{\phi}_\Omega\|_{L^\infty(\partial\Omega)}^2 \leq C \sqrt{\int_{\partial\Omega} |\kappa - \bar{\kappa}|^2 dS}. \quad (20)$$

Indeed if one follows the proof of Theorem 3, one can apply Cauchy-Schwarz on line (52) and bound $\int_{\partial\Omega} p^2 dS$ by CL^2 , thus establishing (20) with $C \sim L^3$. This inequality turns out not to be sufficient to show that the energy (1) decreases along the flow however. Observe that neither inequality can hold without the assumption of connectedness, as the example of two disjoint balls demonstrates.

Remark 3. *In dimensions $d \geq 3$, we expect the above inequality to continue to hold, but are unable to demonstrate it without an assumption that the sets Ω satisfy a positive uniform lower bound on the principal curvatures of the surface $\partial\Omega$. In this case the constant C also depends on this lower bound. Proving Theorem 3 is the only obstacle in extending our results to the case $U = \mathbb{R}^3$. The proof presented in Section 5.1 fails in \mathbb{R}^d for $d \geq 3$ since the Gaussian curvature and mean curvature do not agree.*

Our paper is organized as follows. In Section 2 we set up the appropriate framework for critical points, defining precisely in what sense (8) is satisfied and showing that when $U = \mathbb{R}^2$ critical points are convex when $\bar{\eta}$ is sufficiently small. In Section 3 we introduce the area-preserving curve shortening flow and state some of the main results concerning the flow that we will need. In Section 4 we state precisely the result showing (12), Theorem 10. In Section 5 we establish the necessary inequalities needed to control the behavior of the non-local term in terms of the decay of perimeter along the flow (cf. Theorem 3). Finally we use the geometric inequalities established in Section 5 to prove the above theorems in Section 6 by differentiating the energy (1) along the flow. The counter example (cf. Counter Example 1) will appear at the end of Section 5.

2 The weak Euler-Lagrange equation

In this section we rigorously compute the Euler-Lagrange equation for the class of curves admissible in the sense of Definition 2. We work in \mathbb{R}^2 for simplicity of presentation and hence set $\gamma \equiv 1$. The calculation of the Euler-Lagrange equation is essentially identical on \mathbb{T}^2 however the analysis of critical points differs slightly and so we reserve this for Section 5.2.

In the class of admissible curves (cf. Definition 2) the energy (1) may be written as

$$E(u) = \sum_k \int_0^{L_k} |\gamma'_k(s)| ds + \iint_{\Omega \times \Omega} G(x - y) dx dy, \quad (21)$$

where γ_k is as in Definition 2, and $|\gamma'_k(s)| = 1$ for a.e $s \in [0, 1]$. Since $\gamma_k \cap \gamma_j = \emptyset$ when $j \neq k$, the variations $\gamma_k \mapsto \gamma_k + tv$ such that $\int_0^{L_k} v(s) \cdot (\gamma'_k(s))^\perp ds = 0$ are admissible for $t > 0$ sufficiently small in Definition 1 by letting $\Omega_t = \text{Int}(\gamma_k + tv)$, where $\text{Int}(\gamma)$ denotes the interior of the closed curve γ .

Proposition 4. (*Weak Euler-Lagrange equation*) *Let Ω be a critical point to (21) in the sense of Definition 1, which is admissible in the sense of Definition 2. Then for every k it holds that*

$$\kappa(\gamma_k(s)) + \phi_\Omega(\gamma_k(s)) = \lambda \quad (22)$$

where $\kappa(\gamma_k(s)) = \gamma''_k(s) \cdot (\gamma'_k(s))^\perp$ is the curvature and λ is the Lagrange multiplier from the volume constraint. Moreover $\gamma_k \in C^{3,\alpha}([0, 1])$ for all k .

Proof. By taking variations $t \mapsto \gamma_k + tv$ as described above and differentiating (21) with respect to t , we have

$$\int_0^1 \gamma'_k(s) \cdot v'(s) ds + L_k \int_0^1 \phi_\Omega(\gamma_k(s)) v(s) \cdot (\gamma'_k(s))^\perp ds = 0, \quad (23)$$

for all $v \in W^{1,\infty}([0, 1]; \mathbb{R}^2)$ where we've re-parametrized so that $|\gamma'_k(t)| = L_k$ and $\gamma_k : [0, 1] \rightarrow \mathbb{R}^2$. Equation (23) is the weak Euler-Lagrange equation for (21). Observe that then

$$v \mapsto \int_0^1 \gamma'_k(s) \cdot v'(s) ds,$$

is a bounded linear functional on $W^{1,\infty}([0, 1])$ which extends continuously to a bounded linear functional on $C^0([0, 1])$. Indeed this follows from (23), since $\phi_\Omega(\gamma_k(s)) \in C^{1,\alpha}([0, 1])$ [23] and $\gamma_k \in W^{1,1}([0, 1])$. Thus by the Riesz representation theorem, γ''_k is a finite, vector valued Radon measure on $[0, 1]$. In fact, since $\gamma''_k(s) = (\lambda - L_k \phi_\Omega(\gamma_k(s))) (\gamma'_k(s))^\perp$ as a measure, it follows that $\gamma'_k \in W^{1,1}([0, 1])$. Recalling that $\gamma_k(s) = \gamma_k(0) + L_k \int_0^s e^{i\theta(r)} dr$, we have $\gamma''_k(s) = L_k \theta'(s) e^{i\theta(s)}$ for a.e $s \in [0, 1]$ since $\gamma'_k \in W^{1,1}([0, 1])$ and hence $|\gamma''_k(s)| = L_k |\theta'(s)|$ a.e. This implies $\theta' \in L^1([0, 1])$. Then it holds that the curvature $L_k \kappa(\gamma_k(s)) := \gamma''_k(s) \cdot (\gamma'_k(s))^\perp = L_k \theta'(s)$ is defined a.e $s \in [0, 1]$ and is in $L^1([0, 1])$ with $\kappa(\gamma_k(s)) + \phi_\Omega(\gamma_k(s)) = \lambda$ holding for a.e $s \in [0, 1]$. Then by standard elliptic theory [23], it follows that $\gamma_k \in C^{3,\alpha}([0, 1])$ for $\alpha \in (0, 1)$, implying $C^{3,\alpha}$ regularity of the boundary and that (22) holds strongly for all $s \in [0, 1]$. \square

We then have the following approximation Theorem.

Proposition 5. *Let γ be a closed rectifiable curve with $\theta \in BV([0, 1])$. Then there exists a sequence of C^2 curves γ_n such that*

$$\gamma_n \rightarrow \gamma \in W^{1,1}([0, 1]) \quad (24)$$

$$\theta_n \rightarrow \theta \in L^1([0, 1]) \quad (25)$$

$$\theta'_n \rightharpoonup \theta' \in (C([0, 1])^*, \quad (26)$$

where $(C[0, 1])^*$ is the dual of the space of continuous functions on $[0, 1]$.

Proof. Let $\theta_n := \eta^{1/n} * \theta$ where $\eta^{1/n} := \eta(nx)$ and η is the standard mollifier. Then since $\|\theta\|_{BV([0,1])} < +\infty$ we have

$$\limsup_{n \rightarrow +\infty} \|\theta_n\|_{BV([0,1])} < +\infty. \quad (27)$$

Thus we have $\theta'_n \rightharpoonup \theta'$ in the weak sense of measures and $\theta_n \rightarrow \theta$ in $L^1([0, 1])$ by the embedding $BV([0, 1]) \subset L^1([0, 1])$. The convergence $\gamma_n \rightarrow \gamma$ in $W^{1,1}([0, 1])$ follows immediately. \square

Using the above approximation Theorem we prove that the Gauss-Bonnet theorem continues to hold for closed, rectifiable curves. This is not technically necessary in this section as we have proven that $\partial\Omega$ is always parameterizable by $C^{3,\alpha}$ curves by Proposition 4. However we will use this result in Section 5 to prove the inequalities hold without the assumption of smoothness of the boundary.

Proposition 6. *Let γ be a closed, rectifiable curve with $\theta \in BV([0, 1])$. Then there exists $N \in \mathbb{Z}$ such that*

$$\int_0^1 \theta'(s) ds = \int_{\partial\Omega} \kappa(y) dS(y) = 2\pi N,$$

where N is called the winding number.

Proof. Let γ_n be as in Proposition 5. It follows from the Gauss-Bonnet Theorem for C^2 curves that

$$\int_0^1 \theta'_n(s) ds = 2\pi N_n,$$

for all n where $N_n \in \mathbb{Z}$ must be bounded uniformly in n , since θ_n is uniformly bounded in $BV([0, 1])$. Using Proposition 5 we have $\theta'_n \rightharpoonup \theta'$ weakly in $(C([0, 1])^*$ allowing us to pass to the limit in the above, implying that $N_n = N$ for some $N \in \mathbb{Z}$ for sufficiently large n . Thus $\int_0^1 \theta'(s) ds = \int_{\partial\Omega} \kappa(y) dS(y) = 2\pi N$. \square

Proposition 7. *Let Ω be a critical point of (1) in the sense of Definition 1, admissible in the sense of Definition 2. Then there exists $\bar{\eta}_{cr} > 0$ such that whenever $\bar{\eta} < \bar{\eta}_{cr}$, Ω is convex.*

Proof. Each $\gamma_k \in C^{3,\alpha}([0, 1])$ by Proposition 4. Let γ_k be any interior curve parameterizing $\partial\Omega_k$, ie. the interior of γ_k is contained in the interior of some other curve γ_j for $j \neq k$. By Gauss-Bonnet (cf. Proposition 6) $\int_{\partial\Omega_k} \kappa(y) dS(y) = \frac{2\pi N}{L_k}$ where N is the winding number of the curve γ_k , and (22) we have

$$\kappa(y) + \phi_\Omega(y) = \frac{2\pi N}{L_k} + \bar{\phi}_\Omega \quad (28)$$

holds for $y \in \partial\Omega_k$ where the bar denotes average over $\partial\Omega_k$ and where $N \in \mathbb{Z}$. We wish to show that γ_k is a simple curve, ie. $N = -1$. There is a universal constant $C > 0$ such that

$$|\phi_\Omega(y)| \leq Cm(1 + |\log L|) \text{ when } G(x, y) = -\frac{1}{2\pi} \log |x - y| \quad (29)$$

$$|\phi_\Omega(y)| \leq C \frac{m}{L^\alpha} \text{ when } G(x, y) = \frac{1}{|x - y|^\alpha}.$$

Assume first that $N > 0$. Then for $y \in \partial\Omega_k$ we deduce from (28)–(29) and $L_k \leq L$

$$\kappa(y) \geq \frac{1}{L} (2\pi N - CmL(|\log L| + 1)) \geq \frac{1}{L} (2\pi N - 2\sqrt{\pi}C\bar{\eta}) \text{ when } G(x, y) = -\frac{1}{2\pi} \log |x - y| \quad (30)$$

$$\kappa(y) \geq \frac{1}{L} (2\pi N - CmL^{1-\alpha}) \geq \frac{1}{L} (2\pi N - 2\sqrt{\pi}C\bar{\eta}) \text{ when } G(x, y) = \frac{1}{|x - y|^\alpha}. \quad (31)$$

where the second inequalities follow from the isoperimetric inequality $2\sqrt{\pi}m^{1/2} \leq L$. It is clear that when $\bar{\eta}$ is sufficiently small, $\kappa > 0$ for all points on $\partial\Omega_k$, which is a clear contradiction since γ_k was assumed to be an interior curve. When $N = 0$ then we have once again from (28) and (29)

$$|L\kappa(y)| \leq C\bar{\eta}, \quad (32)$$

for $C > 0$ and all $y \in \partial\Omega_k$. Clearly there is always some $y \in \partial\Omega_k$ such that $\kappa(y) \geq \frac{\pi}{L_k}$. Indeed letting γ_k be a unit speed parametrization of $\partial\Omega_k$, we restrict to $s \in [0, s_0] \subset [0, L_k]$ so that $0 \leq \theta_k(s) \leq 2\pi$. Then $\int_0^{s_1} \theta'_k(s) ds = 2\pi$ and thus by the mean value theorem, there exists an $s \in [0, s_0]$ such that $\theta'_k(s) = \frac{2\pi}{s_0} \geq \frac{\pi}{L_k}$ since $s_0 \in [0, L_k]$. This contradicts (32) for $\bar{\eta}$ sufficiently small. Arguing similarly when $N < 0$, we conclude that in this case, $\kappa < 0$ everywhere when $\bar{\eta}$ is sufficiently small and hence γ_k is simple. Thus we have shown that each γ_k is simple when $\bar{\eta}$ sufficiently small since our estimates do not depend on γ_k . We now show that Ω must in fact be convex.

As before we have

$$\kappa(y) + \phi_\Omega(y) = \frac{2\pi N}{L} + \bar{\phi}_\Omega, \quad (33)$$

where now the average is taken over all of $\partial\Omega$. Since each γ_k is simple, $N \leq 1$ and we claim that in fact $N = 1$. First we show that $N \neq 0$. In this case we have $\bar{\kappa} = 0$ and thus from (28) and (29)

$$|L\kappa(y)| \leq C\bar{\eta}, \quad (34)$$

for $C > 0$ and all $y \in \partial\Omega$. As before, there is always some $y \in \partial\Omega$ such that $\kappa(y) \geq \frac{\pi}{L_1}$, where L_1 denotes the length of outer component of $\partial\Omega$, denoted as $\partial\Omega_1$ (ie. the interior of γ_k is contained in the interior of γ_1 for all k). This is a contradiction of (34) however since $L_1 \leq L$. To see that $N \geq 0$, assume that $N < 0$. Then once again from (28) and (29)

$$\kappa(y) \leq \frac{1}{L}(-2\pi N + CmL(|\log L| + 1)) \leq \frac{1}{L}(-2\pi N + 2\sqrt{\pi}C\bar{\eta}) \text{ when } G(x, y) = -\frac{1}{2\pi} \log |x - y| \quad (35)$$

$$\kappa(y) \leq \frac{1}{L}(-2\pi N + CmL^{1-\alpha}) \leq \frac{1}{L}(-2\pi N + 2\sqrt{\pi}C\bar{\eta}) \text{ when } G(x, y) = \frac{1}{|x - y|^\alpha}. \quad (36)$$

By choosing $\bar{\eta}$ sufficiently small, then we would have $\kappa < 0$ everywhere on $\partial\Omega_1$ which is a contradiction, since γ_1 was assumed to be the exterior curve. Thus $N = 1$ and Ω is simply connected, ie. γ_1 is simple. Then we have by Proposition 6 and (22) again that

$$\kappa(y) \geq \frac{1}{L}(2\pi - C\bar{\eta}) \text{ when } G(x, y) = -\frac{1}{2\pi} \log |x - y| \quad (37)$$

$$\kappa(y) \geq \frac{1}{L}(2\pi - C\bar{\eta}) \text{ when } G(x, y) = \frac{1}{|x - y|^\alpha}, \quad (38)$$

for all $y \in \partial\Omega$, showing that $\kappa > 0$ whenever $\bar{\eta}$ is chosen small enough. Thus Ω is convex when $\bar{\eta}$ is chosen sufficiently small. \square

The assumption that $\bar{\eta}$ be sufficiently small is not simply a technical assumption, as Counter Example 1 demonstrates.

3 Area preserving mean curvature flow

We let Ω be a smooth, compact subset of \mathbb{R}^2 with boundary $\partial\Omega$. Letting X_0 be a local chart of Ω so that

$$X_0 : E \subset \mathbb{R}^2 \rightarrow X_0(E) \subset \partial\Omega \subset \mathbb{R}^2.$$

Then we let $X(x, t)$ be the solution to the evolution problem

$$\begin{aligned} \frac{\partial}{\partial t} X(x, t) &= -(\kappa(t, x) - \bar{\kappa}(t)) \cdot \nu(x, t), \quad x \in E, t \geq 0 \\ X(\cdot, 0) &= X_0, \end{aligned} \quad (39)$$

where $\kappa(t, x)$ is the mean curvature of $\partial\Omega$ at the point x , $\bar{\kappa}(t)$ is the average of the mean curvature on $\partial\Omega_t$:

$$\bar{\kappa}(t) = \oint_{\partial\Omega_t} \kappa(y) dS(y), \quad (40)$$

$\nu(x, t)$ is the normal to $\partial\Omega_t$ at the point x and dS is the 1-dimensional Hausdorff measure. The flow (39) in dimensions $d \geq 3$ was first introduced by Huisken [28] who established existence and asymptotic convergence to round spheres for initially convex domains. The planar version for curves was introduced by Gage [21].

For convenience of notation we define

$$L := |\partial\Omega| \quad (41)$$

$$A := |\Omega|. \quad (42)$$

It is easy to see that the surface area of $\partial\Omega$ is decreasing along the flow. Indeed differentiating the perimeter we have

$$\frac{dL}{dt} = - \int_{\partial\Omega_t} (\kappa - \bar{\kappa})^2 dS. \quad (43)$$

The introduction of the non-local term in (1) will create a term which competes with (43) along the flow (39) as the non-locality favors the spreading of mass. The main element of the proof of Theorem 10 stated in Section 4 will therefore be to show that when the mass is small, the decay in perimeter is sufficient to compensate for the increase in energy of the non-local term in (1) along the flow (39). This is where Theorem 3 will play a crucial role. Before we proceed we must recall some now well known results about the flow (39).

The main result of [21] due to Gage is the following:

Theorem 8. (*Global existence*) *If Ω is convex, then the evolution equation (39) has a smooth solution Ω_t for all times $0 \leq t < \infty$ and the sets Ω_t converge in the C^∞ topology to a round sphere, enclosing the same volume as Ω , exponentially fast, as $t \rightarrow +\infty$.*

In addition there is a local in time existence result in [21]

Theorem 9. (*Local existence*) *If Ω is any smooth embedded set, then there exists a $T > 0$ such that the evolution equation (39) has a smooth solution Ω_t for $t \in [0, T)$.*

Finally we are able to state the main result concerning the flow (39) and the energy (1).

4 Main Results

Our main result, from which the other results follow, is the following.

Theorem 10. (*Energy decrease along the flow*) *Let Ω be admissible in the sense of Definition 2 with $\kappa \in L^2(\partial\Omega)$ and denote Ω_t the evolution of Ω under (39).*

- *If $U = \mathbb{R}^2$ and Ω is convex, then there exists $\bar{\eta}_{cr} > 0$ such that whenever $\bar{\eta} < \bar{\eta}_{cr}$ it holds that*

$$\left. \frac{dE(\Omega_t)}{dt} \right|_{t=0} < 0, \quad (44)$$

with E defined by (1).

- *If $U = \mathbb{T}^2$ and Ω is either convex or homeomorphic to $S_{n=1}$ then there exists an $\bar{\gamma}_{cr} = \bar{\gamma}_{cr} > 0$ such that whenever $\bar{\gamma} < \bar{\gamma}_{cr}$ (44) holds.*

Moreover lower bounds for $\bar{\eta}_{cr}$ and $\bar{\gamma}_{cr}$ are given by

$$\max_{\Omega \subset \mathbb{R}^2} \left(\bar{\eta}^2 \frac{\int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS}{L \|\phi - \bar{\phi}\|_{L^\infty(\partial\Omega)}} \right)^{\frac{1}{2}} \quad \text{and} \quad \max_{\Omega \subset \mathbb{T}^2} \left(\bar{\gamma}^2 \frac{\int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS}{L \|\phi - \bar{\phi}\|_{L^\infty(\partial\Omega)}} \right)^{\frac{1}{2}}$$

respectively, where the maximum is taken over convex sets for the first expression, and over convex sets and sets homeomorphic to $S_{n=1}$ in the latter.

- If Ω is in addition assumed to be smooth, then there exists a constant $a, T > 0$ and $C > 0$ depending only on $\bar{\eta}_{cr}$ (or $\bar{\gamma}_{cr}$) such that

$$\frac{dE(\Omega_t)}{dt} \leq -C \int_{\partial\Omega_t} (\kappa - \bar{\kappa})^2 dS_{\Omega_t} \quad (45)$$

for all $t \in [0, T)$. When $U = \mathbb{R}^2$ and Ω is convex, $T = +\infty$.

Remark 4. From Theorem 11 below, we will see that a lower bound for $\bar{\eta}_{cr}$ is given by $\frac{32}{\pi}$ for $G(x, y) = -\frac{1}{2\pi} \log|x - y|$ and $\frac{8}{\pi} \left(1 + \frac{1}{\pi}\right)^{1-\alpha}$ for $G(x, y) = \frac{1}{|x-y|^\alpha}$. Also a lower bound for $\bar{\gamma}_{cr}$ when $U = \mathbb{T}^2$ is given by a universal constant $C_0 > 0$ depending only on the Green's function for \mathbb{T}^2 .

Using this result we can immediately prove Corollary 1.

PROOF OF COROLLARY 1 Using Proposition 4, we can invoke (45) along with (43) to conclude the desired inequality. \square

5 Geometric inequalities

The main result of this section is a geometric inequality which relates the closeness of connected constant curvature curves and equipotential curves. This will be used in Section 6 to control the non-local term along the flow (39).

Theorem 11. Let $\Omega \subset U$ be convex with $\kappa \in L^2(\partial\Omega)$ and $U = \mathbb{R}^2$ or $U = \mathbb{T}^2$. Then there is a constant $C = C(L, \alpha) > 0$ such that

$$\|\phi_\Omega - \bar{\phi}_\Omega\|_{L^\infty(\partial\Omega)}^2 \leq C \int_{\partial\Omega} |\kappa - \bar{\kappa}|^2 dS. \quad (46)$$

- i) When $U = \mathbb{R}^2$ and $G(x, y) = -\frac{1}{2\pi} \log|x - y|$, the constant can be chosen to be $C = C_{\mathbb{R}^2} = \frac{32AL^3}{\pi} (1 + |\log L|)^2 = \frac{32}{\pi L} \bar{\eta}^2$.
- ii) When $U = \mathbb{R}^2$ and $G(x, y) = \frac{1}{|x-y|^\alpha}$ for $\alpha \in (0, 1)$, the constant can be chosen to be $C = \frac{8A}{\pi} \left(1 + \frac{1}{\pi}\right)^{2(1-\alpha)} L^{3-2\alpha} = \frac{8}{\pi L} \bar{\eta}^2$.
- iii) When $U = \mathbb{T}^2$ the constant can be chosen to be $C = C_{\mathbb{T}^2} = C_0 L^3 (1 + |\log L|^2)^2 = \frac{C_0}{\gamma L} \bar{\gamma}^2$ where $C_0 > 0$ is some universal constant. Moreover the inequality continues to hold for sets homeomorphic to $S_{n=1}$.

We separate the case of $U = \mathbb{R}^2$ and $U = \mathbb{T}^2$ since the ability to estimate the distance to an equipotential surface in terms of the distance to a constant curvature surface (with some abuse of terminology hereafter denoted CMC surface) depends on the types of CMC surfaces which may exist. In \mathbb{R}^2 we first control the isoperimetric deficit in terms of the right hand side of (46), then control the left hand side of (46) by the isoperimetric deficit. This relies crucially on the fact that the ball is the only compact, connected CMC surface in \mathbb{R}^2 . In \mathbb{T}^2 we must account for the possibility of the stripe pattern defined by (11) and need to adapt the inequalities accordingly. Therefore while the approaches are almost identical, separate analysis is needed. The Euclidean case will be treated below in Section 5.1 and the case of the torus will be treated in Section 5.2. We will make repeated use of Bonnesen's inequality [6], which states that given any set simply connected $\Omega \subset \mathbb{R}^2$, it holds that

$$L^2 - 4\pi A \geq \pi^2 (R_{\text{out}}(\Omega) - R_{\text{in}}(\Omega))^2, \quad (47)$$

where R_{in} and R_{out} are

$$\begin{aligned} R_{\text{in}}(\Omega) &= \sup_{B_r \subset \Omega} r \\ R_{\text{out}}(\Omega) &= \inf_{\Omega \subset B_R} R. \end{aligned} \quad (48)$$

See Figure 1 for a diagram showing the various quantities.

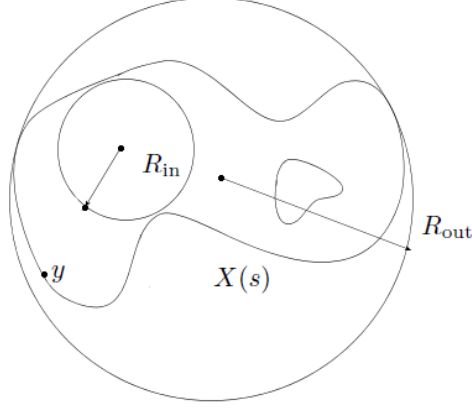


Figure 1: The boundary of Ω is parametrized by $X(s)$ and has inner and outer radii R_{in} and R_{out} .

5.1 The Euclidean case $U = \mathbb{R}^2$

Let $X(s) = (x(s), y(s)) \in \mathbb{R}^2$ be a unit speed parametrization of the curve enclosing Ω , ν the normal to the surface at the point $X(s)$ and $p(s) := \langle X(s), -\nu(s) \rangle$ the support function. We will also use A instead of m for the area to emphasize the geometric nature of the inequalities, although the two are equivalent.

Proposition 12. *Let $\Omega \subset \mathbb{R}^2$ be convex with $\kappa \in L^2(\partial\Omega)$. Then it holds that*

$$L - 2\sqrt{\pi}A^{1/2} \leq \frac{A}{\pi} \int_{\partial\Omega} (\kappa - \bar{\kappa})^2. \quad (49)$$

Proof. Using the generalized Gauss-Green theorem we have

$$A = \int_{\Omega} \det DX(x, y) dx dy = \frac{1}{2} \int_{\partial\Omega} \langle X, -\nu \rangle dS = \frac{1}{2} \int_0^L p(s) ds, \quad (50)$$

where we set $p(s) = \langle X(s), \nu(s) \rangle$ and $\kappa(s) = X''(s) \cdot \nu(s)$. In addition we have

$$\begin{aligned} \int_{\partial\Omega} p\kappa dS &= \int_0^L p(s)\kappa(s) ds = - \int_0^L \langle X(s), \kappa\nu(s) \rangle ds = - \int_0^L \langle X(s), X''(s) \rangle ds \\ &= - \langle X, X' \rangle \Big|_0^L + \int_0^L \langle X'(s), X'(s) \rangle ds \\ &= L. \end{aligned} \quad (51)$$

Adding and subtracting $\bar{\kappa}$ to κ in the integrand on the left side of (51), and using the Gauss-Bonnet theorem for curves (cf. Proposition 6), we find

$$L - \frac{4\pi A}{L} = \int_{\partial\Omega} p(\kappa - \bar{\kappa}) dS. \quad (52)$$

We first prove the inequality when Ω is symmetric about the origin of p . Adding and subtracting \bar{p} from p in (52) we have

$$\begin{aligned} \frac{L^2 - 4\pi A}{L} &\leq \int_{\partial\Omega} (p - \bar{p})(\kappa - \bar{\kappa}) dS \\ &\leq \sqrt{\int_{\partial\Omega} (p - \bar{p})^2 dS} \sqrt{\int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS}, \end{aligned} \quad (53)$$

where we've applied Cauchy-Schwarz on the last line. A simple calculation yields

$$\int (p - \bar{p})^2 = \int p^2 - \frac{4A^2}{L} \quad (54)$$

Using an inequality due to Gage [22] for convex sets symmetric about the origin, we have

$$\int p^2 dS \leq \frac{LA}{\pi}. \quad (55)$$

Inserting (55) into (54) we have

$$\int (p - \bar{p})^2 dS \leq \frac{A}{\pi L} (L^2 - 4\pi A). \quad (56)$$

Inserting the above into (53) we have

$$\sqrt{L^2 - 4\pi A} \leq \sqrt{\frac{AL}{\pi}} \sqrt{\int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS}. \quad (57)$$

Squaring both sides, dividing both sides by L and using $L \leq 2\sqrt{\pi}A^{1/2}$ we have

$$L - 2\sqrt{\pi}A^{1/2} \leq \frac{A}{\pi} \int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS. \quad (58)$$

When Ω is not symmetric about the origin, then choose a point $O \in \Omega$ and any straight line passing through O . Then this line will divide the set Ω into two segments, Ω_1 and Ω_2 . We claim it is always possible to choose this line so that Ω_1 and Ω_2 have the same area. Indeed let $F(\Omega, \theta) := |\Omega_1| - |\Omega_2|$ where θ denotes the angle of the line with respect to some fixed axis. If $F(\Omega, 0) = 0$ then we are done. Otherwise $F(\Omega, 0) > 0$ without loss of generality. But then $F(\Omega, 2\pi) < 0$ and hence by the intermediate value theorem we conclude there exists a $\theta \in (0, 2\pi)$ such that $F(\Omega, \theta) = 0$. Without loss of generality we may orient this line to be parallel to the x axis. Let Ω'_1 and Ω'_2 be the sets formed by reflection across the origin. Then we can apply (49) and we obtain for $i = 1, 2$

$$L'_i - 2\sqrt{\pi}A^{1/2} \leq \frac{A}{\pi} \int_{\partial\Omega'_i} (\kappa'_i - \bar{\kappa}'_i)^2 dS = \frac{A}{\pi} \left(\int_{\partial\Omega'_i} (\kappa'_i)^2 - \frac{4\pi^2}{L'_i} \right).$$

Adding the two inequalities over $i = 1, 2$ and using

$$\frac{1}{L'_1} + \frac{1}{L'_2} \geq \frac{2}{L'_1 + L'_2},$$

we obtain after division by 2

$$L - 2\sqrt{\pi}A^{1/2} \leq \frac{A}{\pi} \int_{\partial\Omega} \kappa^2 dS - \frac{4\pi^2}{L} = \frac{A}{\pi} \int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS,$$

the desired inequality. \square

Next we obtain a quantitative estimate for the closeness to an equipotential surface in terms of the isoperimetric deficit. We will present the proof below for $G(x, y) = -\frac{1}{2\pi} \log |x - y|$ and show how the proof is adapted to the case $G(x, y) = \frac{1}{|x - y|^\alpha}$ in the remark following the proof.

Proposition 13. *Let $\Omega \subset \mathbb{R}^2$ be simply connected. Then there exists a constant $C = C(L, \alpha) > 0$ such that*

$$\|\phi_\Omega - \bar{\phi}_\Omega\|_{L^\infty(\partial\Omega)}^2 \leq C(L^2 - 4\pi A), \quad (59)$$

where ϕ_Ω is defined by (9) for G defined by (13).

- When $G(x, y) = -\frac{1}{2\pi} \log |x - y|$ the constant can be chosen to be $C = 16L^2(1 + |\log L|)^2$.
- When $G(x, y) = K(x - y)$ for $\alpha \in (0, 1)$, the constant can be chosen to be $C = 4\left(1 + \frac{1}{\pi}\right)^{2-2\alpha} L^{2-2\alpha}$.

PROOF OF PROPOSITION 11

Consider any two points $y, z \in \partial\Omega$ and assume first $\phi_\Omega(y) > \phi_\Omega(z)$. Then we have

$$\phi_\Omega(y) - \phi_\Omega(z) = \int_\Omega (G(x, y) + C) - (G(x, z) + C) dx \quad (60)$$

$$\leq \int_{B_{R_{\text{out}}}} G(x, y) + C - \int_{B_{R_{\text{in}}}} G(x, z) + C, \quad (61)$$

where $C = C(R_{\text{out}}) > 0$ is a constant chosen so that $G + C$ is positive on $B_{R_{\text{out}}}$. In particular C can be chosen to be

$$C = C_1 := \max(0, \frac{1}{2\pi} \log 2R_{\text{out}}).$$

Using radial symmetry of the Laplacian equation (61) is in fact equal to

$$\frac{1}{2}(R_{\text{out}}^2 - |y|^2) - \frac{1}{2}(|z|^2 - R_{\text{in}}^2) + C_1(R_{\text{out}}^2 - R_{\text{in}}^2) - \frac{R_{\text{out}}^2}{2} \log R_{\text{out}} + \frac{R_{\text{in}}^2}{2} \log R_{\text{in}} \quad (62)$$

Arguing similarly when $\phi_\Omega(y) \leq \phi_\Omega(z)$ we conclude for all $y, z \in \partial\Omega$ that

$$\begin{aligned} |\phi_\Omega(y) - \phi_\Omega(z)| &\leq (C_1 + 2R_{\text{out}})(R_{\text{out}} - R_{\text{in}}) + \frac{1}{2} |R_{\text{out}}^2 \log R_{\text{out}} - R_{\text{in}}^2 \log R_{\text{in}}| \\ &\leq \left(2 + \frac{1}{\pi}\right) R_{\text{out}}(R_{\text{out}} - R_{\text{in}}) + \frac{1}{2} |R_{\text{out}}^2 \log R_{\text{out}} - R_{\text{in}}^2 \log R_{\text{in}}|. \end{aligned} \quad (63)$$

Observe that the function $f(x) = x^2 \log x$ with $f(0) = 0$ is C^1 and so assuming $x > y$ we have

$$|f(x) - f(y)| \leq |f'|_{L^\infty(0, x)} |x - y| \leq 2|x|(1 + |\log x|)|x - y|, \quad (64)$$

where in the last inequality we've used the fact that $x \mapsto 2|x|(1 + |\log x|)$ is monotone increasing on $(0, +\infty)$. Using Bonnesen's inequality (cf. equation (47)) we have

$$R_{\text{out}} \leq R_{\text{in}} + \frac{1}{\pi} \sqrt{L^2 - 4\pi A} \leq \frac{2L}{\pi} \quad (65)$$

and thus

$$|R_{\text{out}}^2 \log R_{\text{out}} - R_{\text{in}}^2 \log R_{\text{in}}| \leq 2L(|\log L| + 1)(R_{\text{out}} - R_{\text{in}}) \leq 2L(|\log L| + 1)\sqrt{L^2 - 4\pi A} \quad (66)$$

Inserting this into (63) and using (47), we obtain

$$|\phi_\Omega(y) - \phi_\Omega(z)| \leq 4L(1 + |\log L|)\sqrt{L^2 - 4\pi A}. \quad (67)$$

Choosing z so that $|\phi_\Omega(y) - \phi_\Omega(z)| \leq |\phi_\Omega(y) - \bar{\phi}|$ and maximizing over $y \in \partial\Omega$ yields the desired inequality. \square

Remark 5. The above proof easily adapts to the case of the kernel $K(x) = \frac{1}{|x|^\alpha}$. Indeed the constant C_1 above can be taken to be zero, since $K > 0$ everywhere and line (62) can simply be replaced by

$$\left(\int_{B(0,1)} \frac{dx}{|x - y|^\alpha} \right) (R_{\text{out}}^{2-\alpha} - R_{\text{in}}^{2-\alpha}) \leq 2\pi R_{\text{out}}^{1-\alpha} (R_{\text{out}} - R_{\text{in}}) \text{ for } x \in \partial B(0, 1),$$

where we've performed a first order Taylor expansion of the function $f(x) = x^{2-\alpha}$ about the point $x = R_{\text{in}}$. Using Bonnesen's inequality once again we obtain

$$|\phi_\Omega(y) - \phi_\Omega(z)| \leq 2 \left(1 + \frac{1}{\pi}\right)^{1-\alpha} L^{1-\alpha} \sqrt{(L^2 - 4\pi A)}.$$

The rest of the proof is argued identically to the logarithmic case.

PROOF OF THEOREM 11 ITEM I) Now the proof of Theorem 11, items *i*) and *ii*) immediately follows from combining Proposition 13 and Proposition 12 and using $L^2 - 4\pi A \leq 2L(L - 2\sqrt{\pi}A^{1/2})$.

5.2 The torus case $U = \mathbb{T}^2$

We restrict to simply connected sets in $\mathbb{T}^2 = [0, 1]^2$. Any simply connected set lying in \mathbb{T}^2 must either be homeomorphic to the disk or to $S_{n=1}$ defined by (11). In the former case, the analysis is identical to that of the $U = \mathbb{R}^2$ case above. In the latter case, we proceed as follows. By possibly changing coordinates we can see Ω as represented by a curve X with two separate components, $X_{\pm} : [0, L^{\pm}) \rightarrow \mathbb{T}^2$, where L^{\pm} denote the lengths of X_{\pm} , which are unit speed parameterizations of the two components of $\partial\Omega$. We denote S_L as a rectangle of width L and height 1 in the torus. Then we define

$$L_{\text{out}} := \min\{L : \Omega \subset S_L\}, \quad (68)$$

$$L_{\text{in}} := \max\{L : S_L \subset \Omega\}. \quad (69)$$

When X is C^1 , we denote $S_{L_{\text{in}}}$ and $S_{L_{\text{out}}}$ as the two rectangles with widths L_{in} and L_{out} respectively such that X_{\pm} lies in between these two rectangles and touches them tangentially (see Figure 2). The main difficulty lies in controlling the support function p as was done in the proof of Theorem 12 via Gage's inequality (cf. equation (55)). In this case we prove an inequality for the support function when Ω is homeomorphic to $S_{n=1}$ and star shaped (cf. Lemma 1) and then proceed to show that all simply connected critical points must be star shaped (cf. Proposition 14). Finally using the estimate established in Lemma 1, we prove the analogue of Proposition 12 (cf. Proposition 15) for sets homeomorphic to $S_{n=1}$ and subsequently the analogue of Proposition 13 (cf. Proposition 16).

Lemma 1. *Assume $X(s)$ is C^1 and star shaped with respect to the center of $S_{L_{\text{in}}}$. Let p be the support function for $X(s)$ and p^* the support function for $\partial S_{L_{\text{out}}}$ with respect to this center. Then there exists a universal constant $C > 0$ such that*

$$\int_{\partial\Omega} p^2 dS \leq \int_{\partial S_{L_{\text{out}}}} (p^*)^2 dS^* + C(L_{\text{out}} - L_{\text{in}}) + C(L - 2),$$

where dS denotes surface measure on $\partial\Omega$ and dS^* the surface measure on $\partial S_{L_{\text{out}}}$.

Proof. Let O be the center of $S_{L_{\text{in}}}$, $s \in [0, L^{\pm}]$ the arc length parameter for X_{\pm} and $\theta(s)$ the polar angle corresponding to the point $X(s)$ on $\partial\Omega$. Since Ω is star shaped with respect to O , the mapping $s \mapsto \theta(s)$ is a bijection. Therefore we may set $p = p(\theta)$ for $\theta \in (\theta_1, \theta_2)$ and (θ_3, θ_4) which parametrize $X_+(s)$ and $X_-(s)$ respectively. Consider $[0, 1]^2$ as a subset of \mathbb{R}^2 and let $X^*(s)$ be the projection of $X(s)$ onto $\partial S_{L_{\text{out}}}$. Observe that this projection will parametrize a subset of $\partial S_{L_{\text{out}}}$ which includes the two vertical sides (See Figure 2) along with a segment of the horizontal pieces of $\partial S_{L_{\text{out}}}$ lying between $S_{L_{\text{in}}}$ and $S_{L_{\text{out}}}$ on $\partial[0, 1]^2$, and our calculations below will include this segment. To avoid confusion we will denote $\tilde{S}_{L_{\text{out}}}$ and $\tilde{S}_{L_{\text{in}}}$ as the entire rectangles, which includes the top segments along $\{(x, y) : 0 \leq x \leq 1, y = 0 \text{ or } y = 1\} \subset \partial[0, 1]^2$.

Consider a small ray of angular width $\alpha \ll 1$ extending from O and such that $\partial\Omega \neq \partial\Omega^*$ in this ray. Define R_{α} to be the segment of the ray in between the curves $\partial S_{L_{\text{out}}}$ and $\partial\Omega$. Then from Green's theorem we have

$$\int_{R_{\alpha}} X \cdot \nu dS_{R_{\alpha}} = |S_{L_{\text{out}}} \cap R_{\alpha}| - |\Omega \cap R_{\alpha}| := A_{\alpha}. \quad (70)$$

Breaking up the integral on the right, we have

$$\int_{\partial \tilde{S}_{L_{\text{out}}} \cap R_{\alpha}} p^*(\theta) \frac{dS^*}{d\theta} d\theta = \int_{\partial\Omega \cap R_{\alpha}} p(\theta) \frac{dS}{d\theta} d\theta + A_{\alpha}. \quad (71)$$

Now divide by α and send $\alpha \rightarrow 0$. Then we obtain

$$p^*(\theta) \frac{dS^*}{d\theta} = p(\theta) \frac{dS}{d\theta} + \frac{1}{2} [r^*(\theta)^2 - r(\theta)^2], \quad (72)$$

where r^* and r denote the distances to $\partial \tilde{S}_{L_{\text{out}}}$ and $\partial\Omega$ respectively. Then multiplying by p and p^* we have

$$p^*(\theta)p(\theta) \frac{dS^*}{d\theta} = p(\theta)^2 \frac{dS}{d\theta} + \frac{p(\theta)}{2} [r^*(\theta)^2 - r(\theta)^2] \quad (73)$$

and

$$p(\theta)p^*(\theta)\frac{dS}{d\theta} = p^*(\theta)^2\frac{dS^*}{d\theta} + \frac{p^*(\theta)}{2}[[r^*(\theta)]^2 - r(\theta)^2], \quad (74)$$

respectively. Integrating, we have

$$\int_{\theta_1}^{\theta_2} p^2(\theta)\frac{dS}{d\theta}d\theta \stackrel{(73)}{=} \int_{\theta_1}^{\theta_2} p^*(\theta)p(\theta)\frac{dS^*}{d\theta}d\theta - \int_{\theta_1}^{\theta_2} \frac{p(\theta)}{2}[[r^*(\theta)]^2 - r(\theta)^2]d\theta \quad (75)$$

$$\stackrel{(74)}{=} \int_{\theta_1}^{\theta_2} p^*(\theta)^2\frac{dS^*}{d\theta}d\theta + \int_{\theta_1}^{\theta_2} p^*(\theta)p(\theta)\left(\frac{dS^*}{d\theta} - \frac{dS}{d\theta}\right) - \frac{1}{2} \int_{\theta_1}^{\theta_2} [p^*(\theta) - p(\theta)][[r^*(\theta)]^2 - r(\theta)^2]d\theta \quad (76)$$

$$\leq \int_{\theta_1}^{\theta_2} p^*(\theta)^2\frac{dS^*}{d\theta}d\theta + 2 \max_{\theta \in (\theta_1, \theta_2)} p^*(\theta)p(\theta) ((L^+ - 1) + 2(L_{\text{out}} - L_{\text{in}})) + \frac{1}{2} \max_{\theta \in (\theta_1, \theta_2)} [p^*(\theta) - p(\theta)](|S_{L_{\text{out}}}| - A), \quad (77)$$

where L^+ is the length of X_+ and (77) has counted the contribution of $\int_{\theta_1}^{\theta_2} p^*(\theta)p(\theta)\left(\frac{dS^*}{d\theta} - \frac{dS}{d\theta}\right)$ along the vertical pieces of $\partial\tilde{S}_{L_{\text{out}}}$ and the horizontal segment $\partial\tilde{S}_{L_{\text{out}}} \setminus \partial\tilde{S}_{L_{\text{in}}}$. Next we observe that

$$\int_{\theta_3}^{\theta_4} p^*(\theta)^2\frac{dS^*}{d\theta}d\theta + \int_{\theta_1}^{\theta_2} p^*(\theta)^2\frac{dS^*}{d\theta}d\theta = \int_{\partial\tilde{S}_{L_{\text{out}}} \setminus \partial\tilde{S}_{L_{\text{in}}}} (p^*)^2 dS^* + \int_{\partial S_{L_{\text{out}}}} (p^*)^2 dS^* \quad (78)$$

$$\leq 8(L_{\text{out}} - L_{\text{in}}) + \int_{\partial S_{L_{\text{out}}}} (p^*)^2 dS^*. \quad (79)$$

Repeating the above on the curve X_- , using the uniform boundness of p and p^* and $|S_{L_{\text{out}}} - A| \leq |L_{\text{out}} - L_{\text{in}}|$ yields the result. \square

Proposition 14. *For $\bar{\eta}$ sufficiently small, any critical point on \mathbb{T}^2 is star shaped with respect to the center of $S_{L_{\text{in}}}$.*

Proof. Arguing as in Section 2 it is easily seen that the curves X_{\pm} are $C^{3,\alpha}$. Assume the result of the Proposition is false for $X_{\pm}(s)$ and let O be the center of $S_{L_{\text{in}}}$. Then there exists an angle $\theta \in (\theta_1, \theta_2)$ (without loss of generality we choose the curve on the left, $X_+(s)$ in Figure 2) such that $O + (\cos \theta, \sin \theta)$ intersects two points on $X_+(s)$, which we denote as $X_+(s_1)$ and $X_+(s_2)$. Then necessarily $\nu(X_+(s_1)) = -\nu(X_+(s_2))$ for two pairs of such points. Thus, by the mean value theorem it follows that there exists an $s \in [s_1, s_2]$ such that

$$\kappa(X_+(s)) = \frac{\pi}{s_2 - s_1} \geq \frac{\pi}{L}.$$

Then using (22) we have

$$\kappa(X_+(s)) \geq \frac{1}{L}(\pi - L(1 + |\log L|)\gamma C),$$

where $C > 0$ is universal. We thus have

$$\kappa(X_+(s)) \geq \frac{1}{L}(\pi - \bar{\gamma}),$$

since $L \geq 1$. Choosing $\bar{\gamma}$ sufficiently small yields a contradiction of (22) since $|\kappa|L \leq C\gamma L^2(1 + |\log L|) \leq C\bar{\gamma}$. \square

We then have the following version of Proposition 12. We present the proof in the case $A = \frac{1}{2}$ for simplicity of presentation but the proof is easily adapted for any $A \in (0, 1)$.

Proposition 15. *Let $\Omega \subset \mathbb{T}^2$ have a C^2 boundary and be homeomorphic to $S_{n=1}$. Assume in addition that Ω is star shaped with respect to the center of $S_{L_{\text{in}}}$. Then there is a universal constant $C > 0$ such that*

$$|L_{\text{out}} - L_{\text{in}}|^2 \leq C \int_{\partial\Omega} (\kappa - \bar{\kappa})^2. \quad (80)$$

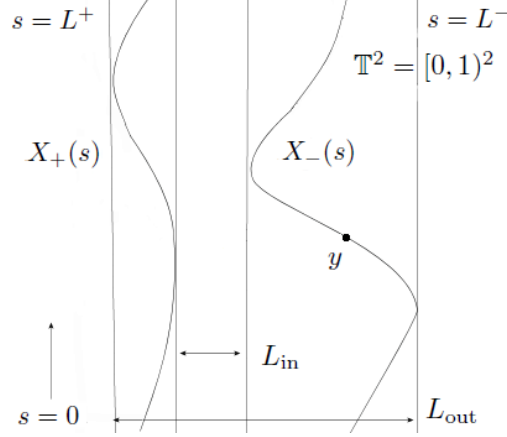


Figure 2: The curve $X(s)$ is composed of two components, $X_{\pm}(s)$ contained in between two rectangles $S_{L_{\text{out}}}$ and $S_{L_{\text{in}}}$ of lengths L_{out} and L_{in} respectively.

Proof. We perform all the analysis on one of the curves X_+ with length L^+ , then argue symmetrically. We point out that below averaged quantities such as \bar{p} and $\bar{\kappa}$ will always denote the average over the entire curve X . As in Proposition 12 we have

$$\begin{aligned} \int_0^{L^+} p\kappa dS &= - \int_0^{L^+} \langle X, \kappa\nu \rangle dS = - \int_0^{L^+} \langle X, X'' \rangle dS \\ &= - \langle X_+, X'_+ \rangle \Big|_0^{L^+} + \int_0^{L^+} \langle X'_+, X'_+ \rangle dS \\ &= L^+ - X'_+(L^+) \cdot X_+(L^+) + X'_+(0) \cdot X_+(0), \end{aligned} \quad (81)$$

where in contrast to Section 5.1, the boundary terms do not vanish. Without loss of generality we choose coordinates so that $X'_+(0) = X'_+(L) = (1, 0)$ (this is possible by periodicity and Rolle's theorem for C^1 functions) and such that $X_+(L^+) - X_-(L^-) = (1, 0)$. Then the above becomes

$$\int p\kappa dS = L^+ - 1. \quad (82)$$

Adding and subtracting $\bar{\kappa}$ to κ in the integrand on the left side of (81) as before, we have

$$L^+ - 1 - \bar{\kappa} \int_0^{L^+} p dS = \int_0^{L^+} p(\kappa - \bar{\kappa}) dS. \quad (83)$$

On each curve X_{\pm} we have

$$\int_0^{L^{\pm}} \kappa = \alpha(L^{\pm}) - \alpha(0),$$

where α denotes the angle of the normal vector with respect to some reference axis. By periodicity $\alpha(0) = \alpha(L^{\pm})$ and so $\bar{\kappa} = 0$. We are left with

$$L^+ - 1 = \int_0^{L^+} (p - \bar{p})(\kappa - \bar{\kappa}). \quad (84)$$

We can compute via the Gauss-Green theorem that $\bar{p} \geq [2A - L_{\text{out}}]/L = [1 - L_{\text{out}}]/L$ since $A = 1/2$, where the term $-L_{\text{out}}/L$ comes from subtracting the contribution from the boundary of the torus. Applying Cauchy-Schwarz to (83) we are left with

$$\int_{\partial\Omega} p^2 - (\bar{p})^2 L \leq \int_{\partial\Omega} p^2 - \frac{(1 - L_{\text{out}})^2}{L}. \quad (85)$$

Using Lemma 1 the above is controlled by

$$C(L_{\text{out}} - L_{\text{in}}) + C(L - 2) + \int_{\partial S_{L_{\text{out}}}} (p^*)^2 dS^* - \frac{(1 - L_{\text{out}})^2}{L}. \quad (86)$$

Observe that the difference between the centers of $S_{L_{\text{in}}}$ and $S_{L_{\text{out}}}$ is controlled by $|L_{\text{out}} - L_{\text{in}}|$. Since the function p^* is defined with respect to the center of $S_{L_{\text{in}}}$, it is easily seen via direct computation, setting $X = X + c - c$ so that $O + c$ is the center of $S_{L_{\text{out}}}$ that

$$\int_{\partial S_{L_{\text{out}}}} (p^*)^2 dS^* \leq \frac{L_{\text{out}}^2}{2} + C(L_{\text{out}} - L_{\text{in}}),$$

where $C > 0$ is universal. Inserting this into (86) we have

$$\begin{aligned} \int_{\partial S_{L_{\text{out}}}} (p^*)^2 dS^* - \frac{(1 - L_{\text{out}})^2}{L} &\leq \frac{2L_{\text{out}}^2 L - 1}{4L} \\ &+ \frac{1}{L} \left(\frac{1}{2} - L_{\text{out}} \right) \leq 2 \frac{L - 2 + L_{\text{out}} - L_{\text{in}}}{L} + C(L_{\text{out}} - L_{\text{in}}). \end{aligned} \quad (87)$$

Thus it follows that

$$\int_{\partial \Omega} (p - \bar{p})^2 dS \leq C(L - 2) + C(L_{\text{out}} - L_{\text{in}}), \quad (88)$$

which when inserted into (84) and after applying Cauchy-Schwarz yields

$$L^+ - 1 \leq C \sqrt{(L - 2) + (L_{\text{out}} - L_{\text{in}})} \sqrt{\int_{\partial \Omega} (\kappa - \bar{\kappa})^2 dS}. \quad (89)$$

Moreover $L - 2 = (L^+ - 1) + (L^- - 1) \geq |L_{\text{out}} - L_{\text{in}}|$ (see Figure 2). Repeating the above analysis on the curve X_- and adding the results, there exists a universal constant $C > 0$ such that

$$L - 2 \leq C \sqrt{L - 2} \sqrt{\int_{\partial \Omega} (\kappa - \bar{\kappa})^2 dS}. \quad (90)$$

Dividing by $\sqrt{L - 2}$ and using $L - 2 \geq |L_{\text{out}} - L_{\text{in}}|$ again we have

$$|L_{\text{out}} - L_{\text{in}}|^{1/2} \leq C \sqrt{\int_{\partial \Omega} (\kappa - \bar{\kappa})^2 dS}, \quad (91)$$

which yields the result upon squaring both sides. □

Now we have the proposition analogous to Proposition 13.

Proposition 16. *Let $\Omega \subset \mathbb{T}^2$ be homeomorphic to $S_{n=1}$. Then there exists a universal constant $C > 0$ such that*

$$\|\phi_\Omega - \bar{\phi}_\Omega\|_{L^\infty(\partial \Omega)} \leq C |L_{\text{out}} - L_{\text{in}}| \quad (92)$$

PROOF OF PROPOSITION 11

Consider any two points $y, z \in \partial \Omega$ and assume first $\phi_\Omega(y) > \phi_\Omega(z)$. Then we have

$$\phi_\Omega(y) - \phi_\Omega(z) = \int_{\Omega} G(x, y) + C - G(x, z) - C dx \quad (93)$$

$$\leq \int_{S_{L_{\text{out}}}} G(x, y) + C - \int_{S_{L_{\text{in}}}} G(x, z) + C, \quad (94)$$

where C is chosen so that $G + C > 0$ on \mathbb{T}^2 . This is possible since G takes the form

$$G(x, y) = -\frac{1}{2\pi} \log |x - y| + S(x - y),$$

where S is smooth [23]. The integrals of the logarithmic terms appearing in (94) represent, up to translations, solutions of the one dimensional Poisson equation

$$-u_{xx} = \begin{cases} 1 & \text{for } 0 \leq x \leq w \\ 0 & \text{for } w \leq x \leq 1 \end{cases} \quad (95)$$

for $w = L_{\text{in}}$ and $w = L_{\text{out}}$. One can explicitly solve (95),

$$u(x) = \begin{cases} -\frac{1}{2}x^2 + (w - \frac{1}{2}w^2)x & \text{for } 0 \leq x \leq w \\ -\frac{1}{2}w^2(x - 1) & \text{for } w \leq x \leq 1 \end{cases} + C_0 \quad (96)$$

where C_0 is a constant. It is then easily seen via direct computation that there is a universal constant $C > 0$ such that the integrals of the logarithmic terms in (94) are controlled by

$$C(L_{\text{out}} - L_{\text{in}}). \quad (97)$$

Since the remaining terms in (94) are bounded in $L^\infty(\mathbb{T}^2)$ we conclude that

$$\phi_\Omega(y) - \phi_\Omega(z) \leq C(L_{\text{out}} - L_{\text{in}}), \quad (98)$$

where $C > 0$ is universal. Arguing identically when $\phi_\Omega(y) \leq \phi_\Omega(z)$ we conclude

$$|\phi_\Omega(y) - \phi_\Omega(z)| \leq C(L_{\text{out}} - L_{\text{in}}).$$

Choosing z so that $|\phi_\Omega(y) - \bar{\phi}_\Omega| \leq |\phi_\Omega(y) - \phi_\Omega(z)|$, and optimizing over $y \in \partial\Omega$ yields the result. \square

PROOF OF THEOREM 11 III) Now the proof of Theorem 11 follows immediately from combining Proposition 15 and Proposition 16 as was done in the case of $U = \mathbb{R}^2$. When Ω is homeomorphic to the disk, one can repeat the analysis for $U = \mathbb{R}^2$. In particular line (63) in the proof of Theorem 11 items i) and ii) will be replaced with

$$|\phi_\Omega(y) - \phi_\Omega(z)| \leq C_0 R_{\text{out}}(R_{\text{out}} - R_{\text{in}}),$$

where the constant C_0 depends only on the Green's potential G for the torus \mathbb{T}^2 . In this case we have a uniform bound on R_{out} , and so the result comes from combining the above estimate with Proposition 12 and using the fact that $L \geq 2$. Otherwise, if Ω is homeomorphic to $S_{n=1}$, the constant comes from multiplying the constants in Proposition 15 and 16.

In the general case when X is not assumed to be C^2 but $\kappa \in L^2(\partial\Omega)$, the result follows by setting $X_\pm = X_\pm(0) + L^\pm \int_0^t e^{i\theta(r)} dr$, mollifying $\theta^\epsilon := \eta^\epsilon * \theta$ and passing to the limit in the inequality, using in particular the fact that $(\theta^\epsilon)' \rightarrow \theta'$ in $L^2([0, 1])$ and $X^\epsilon \rightarrow X$ uniformly, and the fact that the mollified curve will remain star shaped since we mollify θ . We omit the details.

6 Proof of Theorems

PROOF OF THEOREM 10: Let $C(L, \alpha)$ denote the constant appearing in Theorem 11. We prove only the first item when $U = \mathbb{R}^2$ since the second item for $U = \mathbb{T}^2$ is argued identically. Observe that variations $\Omega \rightarrow \Omega_t$ induced by the normal velocity $\kappa - \bar{\kappa}$ are admissible in the sense of Definition 1. Then for any convex set Ω with $\kappa \in L^2(\partial\Omega)$ we have

$$\frac{dE(\Omega_t)}{dt} \Big|_{t=0} = - \int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS - \int_{\partial\Omega} (\kappa - \bar{\kappa})(\phi_\Omega - \bar{\phi}_\Omega) dS \quad (99)$$

$$\leq - \int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS + \sqrt{C(L, \alpha)} L^{1/2} \int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS \quad (100)$$

$$\leq -(1 - \bar{\eta}\tilde{C}) \int_{\partial\Omega} (\kappa - \bar{\kappa})^2 dS, \quad (101)$$

whenever $\bar{\eta} < \bar{\eta}_{cr}$ with $\bar{\eta}_{cr}$ defined as in Theorem 10 and where $\tilde{C} > 0$ is universal. The first line is simply the computation of the derivative of E along the flow (39) at $t = 0$. Line (100) follows from Theorem 11 along with Cauchy Schwarz. This establishes the first part of Theorem 10, observing that (101) implies the lower bound on $\bar{\eta}_{cr}$. When Ω is smooth, there exists $t \in [0, T)$ such that the above holds for all $t \in [0, T)$ by Theorem 9 with $T = +\infty$ when Ω is convex (cf. Theorem 8), establishing the second part of Theorem 10.

PROOF OF THEOREM 1 We have in fact established Theorem 1 from the above calculations. Indeed we have shown that

$$\left. \frac{dE(\Omega_t)}{dt} \right|_{t=0} < 0,$$

where the map $\Omega \mapsto \Omega_t$ is an admissible variation in Definition 1. Therefore by definition Ω is not a critical point if it does not have constant curvature. Using the characterization of compact, connected, constant curvature curves in \mathbb{R}^2 and simply connected constant curvature curves in \mathbb{T}^2 establishes the claim. \square

PROOF OF THEOREM 2 We have shown in Theorem 1 that the only simply connected critical points on \mathbb{T}^2 when $\bar{\gamma} < \bar{\gamma}_{cr}$ are the ball and the stripe pattern $S_{n=1}$. By the result of Sternberg and Topaloglu [46] $E(\Omega) \geq E(S_{n=1})$ when $\bar{\gamma} < \bar{\gamma}_{cr}$ by possibly lowering $\bar{\gamma}_{cr}$. Moreover since any minimizer has a reduced boundary with $C^{3,\alpha}$ regularity [46], we have from (45) that there is a $T > 0$ so that

$$\frac{dE(\Omega_t)}{dt} \leq -C \int_{\partial\Omega_t} (\kappa - \bar{\kappa})^2 dS, \quad (102)$$

for all $t \in [0, T)$ and where $C = C(\bar{\gamma}) > 0$. Integrating and using $E(\Omega) \geq E(S_{n=1})$ when $\bar{\gamma} < \bar{\gamma}_{cr}$ we have

$$E(\Omega) \geq E(S_{n=1}) + C \int_0^T \int_{\partial\Omega_t} (\kappa - \bar{\kappa})^2 dS =: E(S_{n=1}) + F(\Omega) \quad (103)$$

where $F(\Omega)$ is defined implicitly from the above. The result holds for the non-rescaled quantity γ since we have the a-priori bound on L coming from $L \leq \min_{\mathcal{A}_{1/2}} E$. \square

COUNTER EXAMPLE 1

We consider annuli of radii $r, R > 0$ with $r < R$ and let $\Omega = \{x : r \leq |x| \leq R\}$. Then we can explicitly calculate ϕ_Ω , $\bar{\phi}_\Omega$, κ and $\bar{\kappa}$ on $\partial\Omega$ for both the kernels K and $-\frac{1}{2\pi} \log |x|$. We consider first the logarithmic case.

$$\text{Case 1: } G(x, y) = -\frac{1}{2\pi} \log |x - y|$$

Solving Poisson's equation $-\Delta\phi_\Omega = 1_\Omega$ explicitly in radial coordinates we obtain

$$\phi_\Omega(x) = \frac{1}{2}(R^2 - |x|^2) - \frac{1}{2}(|x|^2 - r^2) - \frac{R^2}{2} \log R + \frac{r^2}{2} \log r \text{ for } x \in [r, R]. \quad (104)$$

It is easily seen that $\bar{\kappa} = 0$ with $\kappa = \frac{-1}{r}$ on ∂B_r and $\kappa = \frac{1}{R}$ on ∂B_R . We first demonstrate Counter Example 1. We wish to find $R > r > 0$ such that

$$\kappa(x) + \phi_\Omega(x) = \kappa(y) + \phi_\Omega(y),$$

for $x \in \partial B_r$, $y \in \partial B_R$. We claim this is the case. Indeed setting $R = 2r$ with $r = (\frac{1}{2})^{1/3}$ we have $\bar{\eta} := (\frac{1}{2})^{1/3} (3\pi)^{1/2} (1 + \log(6\pi 2^{-1/3}))$. Then with these choices of r and R ,

$$-\frac{1}{r} + \frac{1}{2}(R^2 - r^2) = \frac{1}{R} + \frac{1}{2}(r^2 - R^2). \quad (105)$$

holds. Thus Ω is a solution to (8) for these choices of r and R .

Case 2: $G(x, y) = \frac{1}{|x-y|^\alpha}$

In this case we have

$$\phi_\Omega(x) = R^{2-\alpha} \int_{B(0,1)} \frac{d\bar{y}}{|x/R - \bar{y}|^\alpha} - r^{2-\alpha} \int_{B(0,1)} \frac{d\bar{y}}{|x/r - \bar{y}|^\alpha} \quad (106)$$

$$= \left(\int_{B(0,1)} \frac{dy}{|\tilde{x} - y|^\alpha} \right) (R^{2-\alpha} - r^{2-\alpha}) \text{ for } \tilde{x} \in \partial B(0,1). \quad (107)$$

As in the previous case, it is seen via direct computation that Ω is a solution to (8) when $R = 2r$ and $\bar{\eta} = m^{1/2} L^{2-\alpha} = C_0$ where C_0 is an explicit constant, thus establishing Counter Example 1 for $G = K$.

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References

- [1] G. Alberti, R. Choksi, and F. Otto. Uniform energy distribution for an isoperimetric problem with long-range interactions. *J. Amer. Math. Soc.*, 22:569–605, 2009.
- [2] A. D. Alexandrov. A characteristic property of spheres. *Ann. Mat. Pura Appl.*, 4:303–315, 1962.
- [3] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, New York, 2000.
- [4] L. Ambrosio, V. Caselles, S. Masnou, and J.M Morel *Connected components of sets of finite perimeter with applications to image processing*. *J. Eur. Math. Soc.*, 3:39-92, 2001
- [5] D. Antonopoulou, G. Karali, and I.M. Sigal. Stability of spheres under volume-preserving mean curvature flow. *J. Dynamics of PDE*, 7:327-344, 2010.
- [6] T. Bonnesen. Sur une amélioration de l’inégalité isopérimétrique du cercle et la démonstration d’une inégalité de Minkowski *C. R. Acad. Sci*, 172:1087-1089, 1921
- [7] L. Q. Chen and A. G. Khachaturyan. Dynamics of simultaneous ordering and phase separation and effect of long-range Coulomb interactions. *Phys. Rev. Lett.*, 70:1477–1480, 1993.
- [8] R. Choksi. Scaling laws in microphase separation of diblock copolymers. *J. Nonlinear Sci.*, 11:223–236, 2001.
- [9] R. Choksi, S. Conti, R. V. Kohn, and F. Otto. Ground state energy scaling laws during the onset and destruction of the intermediate state in a Type-I superconductor. *Comm. Pure Appl. Math.*, 61:595–626, 2008.
- [10] R. Choksi, R. V. Kohn, and F. Otto. Energy minimization and flux domain structure in the intermediate state of a Type-I superconductor. *J. Nonlinear Sci.*, 14:119–171, 2004.
- [11] R. Choksi, M. Maras, and J. F. Williams. 2D phase diagram for minimizers of a Cahn–Hilliard functional with long-range interactions. *SIAM J. Appl. Dyn. Syst.*, 10:1344–1362, 2011.
- [12] R. Choksi and L. A. Peletier. Small volume fraction limit of the diblock copolymer problem: I. Sharp interface functional. *SIAM J. Math. Anal.*, 42:1334–1370, 2010.
- [13] R. Choksi and M. A. Peletier. Small volume fraction limit of the diblock copolymer problem: II. Diffuse interface functional. *SIAM J. Math. Anal.*, 43:739–763, 2011.

- [14] R. Choksi and P. Sternberg. On the first and second variations of a non-local isoperimetric problem. *J. Reine angew. Math.*, 611:75–108, 2007.
- [15] P. G. de Gennes. Effect of cross-links on a mixture of polymers. *J. de Physique – Lett.*, 40:69–72, 1979.
- [16] E. De Giorgi, Sulla proprietà isoperimetrica dell’ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita. *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8)*, 5:33–44, 1958.
- [17] V. J. Emery and S. A. Kivelson. Frustrated electronic phase-separation and high-temperature superconductors. *Physica C*, 209:597–621, 1993.
- [18] G. Friesecke, R. D. James, and S. Müller. A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence. *Arch. Ration. Mech. Anal.*, 180:183–236, 2006.
- [19] L.E Fraenkel. *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*. Cambridge university press, Cambridge, 2000.
- [20] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math.*, 168:941–980, 2008.
- [21] M. Gage. On an area-preserving evolution equation for plane curves, *Contemp. Math.*, 51:51–62 1986.
- [22] M. Gage. An isoperimetric inequality with applications to curve shortening, *Duke Math. J.*, 50:1225–1229 1983.
- [23] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 1983.
- [24] S. Glotzer, E. A. Di Marzio, and M. Muthukumar. Reaction-controlled morphology of phase-separating mixtures. *Phys. Rev. Lett.*, 74:2034–2037, 1995.
- [25] D. Goldman, C. B. Muratov, and S. Serfaty. The Γ -limit of the two-dimensional Ohta-Kawasaki energy. II. Droplet arrangement via the renormalized energy. (in preparation).
- [26] D. Goldman, C. B. Muratov, and S. Serfaty. The Γ -limit of the two-dimensional Ohta-Kawasaki energy. I. Droplet density (Preprint: <http://www.math.nyu.edu/~dgoldman/gamma2dokzero.pdf>).
- [27] D. Goldman, A. Volkmann A short note on the regularity of critical points to the Ohta-Kawasaki energy. (Preprint).
- [28] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20:237–266, 1984.
- [29] R. V. Kohn. Energy-driven pattern formation. In *International Congress of Mathematicians. Vol. I*, pages 359–383. Eur. Math. Soc., Zürich, 2007.
- [30] S. Lundqvist and N. H. March, editors. *Theory of inhomogeneous electron gas*. Plenum Press, New York, 1983.
- [31] U. Massari Esistenza e regolarità delle ipersurface di curvatura media assegnata in \mathbb{R}^n *Arch. Rat. Mech. Anal.*, 55:357–382, 1974
- [32] C. B. Muratov. *Theory of domain patterns in systems with long-range interactions of Coulombic type*. Ph. D. Thesis, Boston University, 1998.
- [33] C. B. Muratov. Theory of domain patterns in systems with long-range interactions of Coulomb type. *Phys. Rev. E*, 66:066108 pp. 1–25, 2002.
- [34] C. B. Muratov. Droplet phases in non-local Ginzburg-Landau models with Coulomb repulsion in two dimensions. *Comm. Math. Phys.*, 299:45–87, 2010.

- [35] H. Knupfer C. B. Muratov, On an isoperimetric problem with a competing non-local term. I. The planar case. Preprint available at <http://arxiv.org/abs/1109.2192>.
- [36] H. Knupfer C. B. Muratov. On an isoperimetric problem with a competing non-local term. II. The higher dimensional case. Preprint.
- [37] M. Muthukumar, C. K. Ober, and E. L. Thomas. Competing interactions and levels of ordering in self-organizing polymeric materials. *Science*, 277:1225–1232, 1997.
- [38] E. L. Nagaev. Phase separation in high-temperature superconductors and related magnetic systems. *Phys. Uspekhi*, 38:497–521, 1995.
- [39] T. Ohta and K. Kawasaki. Equilibrium morphologies of block copolymer melts. *Macromolecules*, 19:2621–2632, 1986.
- [40] W. Reichel. Characterization of balls by Riesz-Potentials. *Annali di Matematica Pura ed Applicata*, 188:235–245, 2009.
- [41] X. Ren and J. Wei. Many droplet pattern in the cylindrical phase of diblock copolymer morphology. *Rev. Math. Phys.*, 19:879–921, 2007.
- [42] X. F. Ren and J. C. Wei. On the multiplicity of solutions of two nonlocal variational problems. *SIAM J. Math. Anal.*, 31:909–924, 2000.
- [43] M. Seul and D. Andelman. Domain shapes and patterns: the phenomenology of modulated phases. *Science*, 267:476–483, 1995.
- [44] L. Simon. *Lectures on geometric measure theory*. Australian National University, 1983.
- [45] E. Spadaro. Uniform energy and density distribution: diblock copolymers’ functional. *Interfaces Free Bound.*, 11:447–474, 2009.
- [46] P. Sternberg and I. Topaloglu On the global minimizers to a non-local isoperimetric problem in two dimensions. *Interfaces Free Bound.*, 13:155–169, 2011.
- [47] R. Xiaofeng and W. Juncheng A toroidal tube solution to a problem involving mean curvature and Newtonian potential. *Interfaces Free Bound.*, 13:127–154, 2011.